W10. (Solution by the proposer.) Without loss of generality we can assume that $a \geq b \geq c$ from which immediately follows that $a+b \geq a+c \geq b+c$ and

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

Since the first and the last sequences are sorted in the same way, by applying rearrangement inequality, we get

$$\begin{aligned}
3n\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &\geq 3n\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\\
3n\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &\geq 3n\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)\\
\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &\geq \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}
\end{aligned}$$

Adding up the preceding inequalities, yields

$$(6n+1)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 9n+\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$$

from which we obtain

$$\frac{1}{3}\left(\frac{(6n+1)a-b}{b+c}+\frac{(6n+1)b-c}{c+a}+\frac{(6n+1)c-a}{a+b}\right) \geq 3n$$

Taking into account AM-QM inequality, we have

$$\sqrt{\frac{1}{3}\left[\left(\frac{(6n+1)a-b}{b+c}\right)^2+\left(\frac{(6n+1)b-c}{c+a}\right)^2+\left(\frac{(6n+1)c-a}{a+b}\right)^2\right]}$$

$$\geq \frac{1}{3}\left(\frac{(6n+1)a-b}{b+c}+\frac{(6n+1)b-c}{c+a}+\frac{(6n+1)c-a}{a+b}\right) \geq 3n$$

Squaring and arranging terms, yields

$$\left(\frac{(6n+1)a-b}{b+c}\right)^2+\left(\frac{(6n+1)b-c}{c+a}\right)^2+\left(\frac{(6n+1)c-a}{a+b}\right)^2 \geq 27n^2$$

from which the statement follows. Equality holds when $a=b=c$ and we are done.

Second solution. Let $m := 3n$.Then

$$\sum_{cyc}\left(\frac{(6n+1)\,a-b}{n\,(b+c)}\right)^2 \geq 27 \iff \sum_{cyc}\left(\frac{(6n+1)\,a-b}{b+c}\right)^2 \geq 27n^2 \iff$$

$$\iff \sum_{cyc}\left(\frac{(2m+1)\,a-b}{b+c}\right)^2 \geq 3m^2 \qquad (1)$$

and we will prove that inequality **(1)** holds for any $m \in \mathbb{N}$. Since

$$\frac{((2m+1)\,a-b)^2}{b+c} \geq 2m\,((2m+1)\,a-b) - m^2\,(b+c) \iff$$

$$\iff ((2m+1)\,a-b-m\,(b+c))^2 \geq 0$$

then

$$\sum_{cyc}\left(\frac{(2m+1)\,a-b}{b+c}\right)^2 \geq \sum_{cyc}\left(\frac{2m\,((2m+1)\,a-b)}{b+c} - m^2\right).$$

Thus, suffices to prove inequality

$$\sum_{cyc}\frac{2m\,((2m+1)\,a-b)}{b+c} \geq 6m^2 \iff \sum_{cyc}\frac{(2m+1)\,a-b}{b+c} \geq 3m \qquad (2)$$

Inequality (2) holds because by Cauchy Inequality

$$\sum_{cyc}\frac{(2m+1)\,a-b}{b+c} = \sum_{cyc}\frac{((2m+1)\,a-b)^2}{(b+c)\,((2m+1)\,a-b)} \geq$$

$$\geq \frac{\left(\sum_{cyc}((2m+1)\,a-b)\right)^2}{\sum_{cyc}(b+c)\,((2m+1)\,a-b)} = \frac{4m^2\,(a+b+c)^2}{(4m+1)\,(ab+ac+bc)-(a^2+b^2+c^2)}$$

and

$$\frac{4m^2\,(a+b+c)^2}{(4m+1)\,(ab+ac+bc)-(a^2+b^2+c^2)} \geq 3m \iff$$

$$\iff 4m\,(a+b+c)^2 \geq$$

$$\geq 3(4m+1)(ab+ac+bc) - 3\left(a^2+b^2+c^2\right) \iff$$

$$4m(a+b+c)^2 \geq$$

$$\geq 3(4m+1)(ab+ac+bc) - 3(a+b+c)^2 + 6(ab+ac+bc) \iff$$

$$(4m+3)(a+b+c)^2 \geq 3(4m+3)(ab+ac+bc) \iff (a+b+c)^2 \geq$$

$$\geq 3(ab+ac+bc).$$

Arkady Alt

Third solution. Using the inequality, we have

$$x^2+y^2+z^2 \geq \frac{1}{3}(x+y+z)^2$$

$$\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 + \left(\frac{(6n+1)b-c}{n(c+a)}\right)^2 + \left(\frac{(6n+1)c-a}{n(a+b)}\right)^2 \geq$$

$$\geq \frac{1}{3}\left(\frac{(6n+1)a-b}{n(b+c)} + \frac{(6n+1)b-c}{n(c+a)} + \frac{(6n+1)c-a}{n(a+b)}\right)^2$$

Using Cauchy-Buniakowski-Schwarz inequality, we get

$$\frac{(6n+1)a-b}{n(b+c)} + \frac{(6n+1)b-c}{n(c+a)} + \frac{(6n+1)c-a}{n(a+b)} \geq$$

$$\geq (CBS)\left[\frac{\left(\sum(6n+1)a - \sum a\right)^2}{\sum((6n+1)a-b)(n(b+c))}\right] = \frac{36n^2\left(\sum a\right)^2}{n\left((12n+1)\sum ab - \sum a^2\right)}$$

We show that

$$\frac{36n^2\left(\sum a\right)^2}{n\left((12n+1)\sum ab - \sum a^2\right)} \geq 9$$

$$\frac{36n^2\left(\sum a\right)^2}{n\left((12n+1)\sum ab - \sum a^2\right)} \geq 9 \Leftrightarrow$$

$$\Leftrightarrow 4n\left(\sum a\right)^2 \geq (12n+1)\sum ab - \sum a^2 \Rightarrow (4n+1)\left(\sum a^2 - \sum ab\right) \geq 0$$

that is true.

Nicuşor Zlota

W11. (**Solution by the proposer.**) First, we observe that there are numbers satisfying the equality stated, for instance, 1, 2, 4, and 5. To prove the statement we argue by contradiction. Suppose that there exists four positive integers x, y, z, t with $x > y > z > t$ such that $x^2 + xz - z^2 = y^2 + yt - t^2$ and $p = xy + zt$ is prime. Substituting $x = \dfrac{p - zt}{y}$ in the expression $x^2 + xz - z^2 = y^2 + yt - t^2$, we get

$$\left(\frac{p - zt}{y}\right)^2 + \left(\frac{p - zt}{y}\right)z - z^2 = y^2 + yt - t^2$$

and reordering terms, yields

$$p(p - 2zt + yz) = (y^2 + z^2)(y^2 + yt - t^2)$$

Since p is prime, then p divides $y^2 + z^2$ or divides $y^2 + yt - t^2$.

- If $p \mid y^2 + z^2$, then $0 < y^2 + z^2 < 2xy < 2(xy + zt) = 2p$ and this implies $y^2 + z^2 = p = xy + zt$ from which follows $y \mid z(z - t)$. Since $xy + zt$ is prime, then $\gcd(y, z) = 1$, and therefore, $p \mid (z - t)$ which is impossible because $0 < z - t < z < y$.

- If $p \mid y^2 + yt - t^2$, then $0 < y^2 + yt - t^2 < 2(xy + zt) = 2p$ and this implies $y^2 + yt - t^2 = p$. That is, $xy + zt = y^2 + yt - t^2 = x^2 - xz - z^2$ from which follows $x \mid z(z + t)$ and $y \mid t(z + t)$. As $\gcd(xy, zt) = 1$, then $x \mid (z + t)$ and $y \mid (z + t)$. Since $0 < z + t < 2x$ and $0 < z + t < 2y$, then $z + t = x$ and $z + t = y$ which is impossible.

The preceding contradictions let us to conclude that $xy + zt$ is composite.

W12. (**Solution by the proposer.**) The limit equals $4e^{\gamma}$ where γ denotes the Euler-Mascheroni constant. A calculation shows that

$$O_n = \gamma_{2n} - \frac{1}{2}\gamma_n + \ln 2 + \ln\sqrt{n},$$